Journal of Engineering Mathematics, Vol. 10, No. 4, October 1976
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Vibrations of a rotating flexible rod clamped off the axis of rotation

W. D. LAKIN

Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1

(Received July 24, 1975)

SUMMARY
We consider a fourth-order boundary value problem associated with the small vibrations of a uniform flexible rod which is clamped at one end and rotates in a plane perpendicular to the axis of rotation. A significant feature is that the axis of rotation does not pass through the clamped end itself. For rapid rotation rates, the governing equation involves a small parameter and must be treated by singular perturbation techniques. A second parameter fixes the relative location of two turning points. For a range of this second parameter, consistent approximations to the characteristic equation are derived, and the limiting behavior of the eigenvalues is obtained.

1. Introduction

In this paper, we consider a boundary-value problem arising from the transverse vibrations of a flexible rod which is clamped at one end and rotates with constant angular velocity in a plane perpendicular to the axis of rotation. The rod is assumed not to twist and to have uniform density and cross-section. Eigenvalues of the boundary-value problem determine the rod's natural frequencies of vibration.

Rotating flexible rods have been extensively studied in the past. However, most treatments ([2], [4], and [5]) assume that the rod is hub-clamped, i.e. the axis of rotation passes through the rod's clamped end. In the present work, we assume the clamped end is off the axis of rotation, and hence describes a circle of radius $R > 0$ as the rod rotates. A typical example involves a flexible rod which is fixed to the rim of a steadily rotating wheel and extends inward toward the hub like a partial spoke. This situation is shown in Figure 1. For rotating rods clamped off the axis of rotation, only the static buckling problem has previously been examined. Mostaghel and Tadjbakhsh [7] have used numerical procedures to determine the critical rotation rate for buckling. An improved estimate has also been obtained by Nachman [8] using perturbation techniques.

2. The basic equation

To treat the present vibration problem, let s denote arc length along the rod measured from the clamped end $s = 0$, and let $u(s, t)$ be the transverse displacement of the rod. The vibrations will be assumed sufficiently small so that non-linear terms may be consistently neglected. If the rod has length L, cross-sectional area A, mass per unit volume ρ, bending stiffness EI, and rotates with constant angular velocity Ω, the partial differential

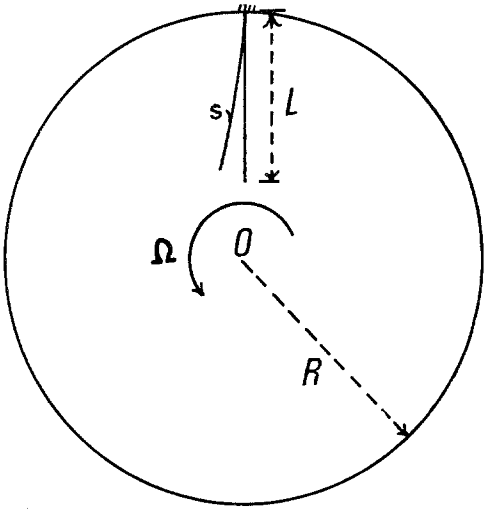

Figure 1. Flexible rod fixed to rim of a steadily rotating wheel.

equation governing the vibrations can be written as

$$EI\frac{\partial^4 u}{\partial s^4} - \frac{\partial}{\partial s}\left\{P(s)\frac{\partial u}{\partial s}\right\} = -\rho A\frac{\partial^2 u}{\partial t^2}. \tag{2.1}$$

In this equation, the function

$$P(s) = \rho A\Omega^2(L-s)[\tfrac{1}{2}(L+s) - R] \tag{2.2}$$

is the total tensile force at position s. At the clamped end of the rod ($s = 0$), we must have zero displacement and slope, while at the free end ($s = L$) both shear force and bending moment must vanish. Hence,

$$\left.\begin{aligned} &u(0,t) = \frac{\partial u}{\partial s}(0,t) = 0 \\ &\frac{\partial^2 u}{\partial s^2}(L,t) = \frac{\partial^3 u}{\partial s^3}(L,t) = 0 \end{aligned}\right\} \tag{2.3}$$

We will seek periodic solutions $u(s,t) = w(s)e^{i\omega t}$. The system (2.1) and (2.3) may now be put in a form more suitable for mathematical analysis by introducing the dimensionless variable

$$x = -s/L \tag{2.4}$$

and the dimensionless ratios

$$\tilde{\varepsilon}^3 = \frac{EI}{\rho A\Omega^2 L^4},\ \alpha = R/L,\ \text{and}\ \lambda = \left(\frac{\omega}{\Omega}\right)^2. \tag{2.5}$$

With these scalings, the governing equation becomes

$$\tilde{\varepsilon}^3 w^{\text{iv}} - \tfrac{1}{2}(1+x)(1-2\alpha-x)w'' + (x+\alpha)w' - \lambda w = 0, \tag{2.6}$$

and the associated boundary conditions are now

$$w(0) = w'(0) = 0 \text{ and } w''(-1) = w'''(-1) = 0. \tag{2.7}$$

Throughout this work, we will assume rapid rotation so that $\tilde{\varepsilon} \ll 1$. From (2.5), we also note that the parameter α and the eigenvalue λ will both be real and positive.

The reduced equation obtained by formally setting $\tilde{\varepsilon}$ to zero in (2.6) is only of second order. Hence, for small $\tilde{\varepsilon}$, to derive asymptotic approximations to solutions of (2.6), singular perturbation methods must be used. It is convenient to regard x as a complex variable. For the boundary value problem, we then desire approximations valid in bounded regions of the complex plane which contain the real interval $[-1, 0]$.

The distinctive character of the governing differential equation (2.6) comes from the two turning points on the real axis at $x_1 = -1$ and $x_2 = 1 - 2\alpha$ where the coefficient of the second derivative w'' vanishes. A novel feature here is that by varying the value of α, the position of the second turning point x_2 may be shifted. For example, when $\alpha = 1$, the two simple turning points x_1 and x_2 coalesce to form a single higher order turning point at x_1. In the boundary value problem, however, regardless of the size of α, there is always a turning point directly at the left hand endpoint. Outer expansions alone are thus not adequate for formation of a characteristic equation for the eigenvalue λ. Compatable inner expansions valid at and near x_1 must also be used.

In the remainder of this work, we will obtain asymptotic approximations to the eigenvalues of the boundary value problem for α in the range

$$0 < \alpha < \tfrac{1}{2}. \tag{2.8}$$

In this case, the second turning point x_2 lies outside of the region of interest, but the boundary point $x_0 = 0$ lies directly on a Stokes line in the complex plane. This range of α corresponds to rods with L order one but R relatively small. If we allow R to be non-negative rather than strictly positive, then the limiting case of the hub-clamped rod $\alpha = 0$ may also be included in (2.8) without difficulty. Indeed, one physical example for values of α in the lower end of this range is a wobbling hub-clamped rod. Specifically, consider a flexible rod rotating in the horizontal plane which is clamped at one end to a vertical driving shaft. If the shaft is either slightly bent or otherwise out of true vertical, but the rod's plane of rotation remains horizontal, then the axis of rotation will pass through a point 0 along the length of the rod rather than through the clamped end. Viewed from above, the clamped end will move in a small circle having 0 as its center.

3. Transformation and solution

For α in the range (2.8), equation (2.6) may consistently be transformed to a standard form which does not involve α. We define new independent and dependent variables y and $\phi(y)$ through the relations

$$y = \frac{x + \alpha}{1 - \alpha} \text{ and } \phi(y) = w(x). \tag{3.1}$$

The two turning points x_1 and x_2 are thus mapped onto $y_1 = -1$ and $y_2 = +1$, respectively, and equation (2.6) becomes

$$\varepsilon^3\phi^{iv} - \tfrac{1}{2}(1-y^2)\phi'' + y\phi' - \lambda\phi = 0 \tag{3.2}$$

with

$$\varepsilon^3 = \frac{\tilde{\varepsilon}^3}{(1-\alpha)^4} = \frac{EI}{\rho A\Omega^2(L-R)^4}. \tag{3.3}$$

This equation is associated with a hub-clamped rod which rotates with the same constant angular velocity Ω but has length $L - R$.

Using (3.1), the right hand endpoint in the present boundary value problem moves away from the origin in the x-plane into the open real interval (0, 1) in the y-plane. We now have $y_0 = y(x_0) = \alpha(1-\alpha)^{-1}$, and the boundary conditions (2.7) become

$$\phi\left(\frac{\alpha}{1-\alpha}\right) = \phi'\left(\frac{\alpha}{1-\alpha}\right) = \phi''(-1) = \phi'''(-1) = 0. \tag{3.4}$$

For the hub-clamped rod, the first two conditions in (3.4) would be evaluated at the origin allowing simplifications in the eigenvalue relation which are not possible when α is non-zero.

To form a characteristic equation for the eigenvalues of (3.2, 4), we require approximations to a set of four linearly independent exact solutions of (3.2) which are "numerically satisfactory" in the sense of Miller [6]. Although the actual forms of these exact solutions are, of course, unknown, the solutions may be defined to within multiplicative constants by their asymptotic properties as follows:

(i) The solution $\phi_0(y)$ is well-balanced in bounded domains containing the real interval $[-1, \alpha/(1-\alpha)]$. In particular, this solution is analytic at the turning point $y_1 = -1$.

(ii) The solution $\phi_1(y)$ is purely balanced in the sector $0 > \mathrm{ph}(y+1) > -2\pi/3$ of the complex plane bounded by Stokes lines. Care must be taken with this solution as adding arbitrary multiples of ϕ_0 to ϕ_1 will not alter the stated asymptotic behavior.

(iii) The two solutions $\chi_1(y)$ and $\chi_2(y)$ are recessive in the sectors $|\mathrm{ph}(y+1)| < \pi/3$ and $-\pi/3 > \mathrm{ph}(y+1) > -\pi$, respectively, of the complex plane bounded by anti-Stokes lines.

Approximations to these four exact solutions have been derived by Lakin [4] and are given here in a modified form more suitable for use at non-zero values of y_0. The four required outer expansions will be denoted by $\bar{\phi}_0$, $\bar{\phi}_1$, $\bar{\chi}_1$, and $\bar{\chi}_2$, the four inner expansions by $\tilde{\phi}_0$, $\tilde{\phi}_1$, $\tilde{\chi}_1$, and $\tilde{\chi}_2$.

Partial sums of the outer expansions $\bar{\chi}_1$ and $\bar{\chi}_2$ may be obtained by the WKBJ technique, and are best expressed in terms of the Langer variable

$$\eta = \tfrac{1}{2}\left\{3\int_{-1}^{y}(1-y^2)^{\frac{1}{2}}dy\right\}^{\frac{2}{3}}. \tag{3.5}$$

The turning point $y_1 = -1$ now corresponds to $\eta_1 = 0$, and $\eta'(-1) = 1$. When used in (3.2), this variable explicitly brings out the turning point nature of the equation. If

$$\bar{\chi}_{\pm}(\eta) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\varepsilon^{\frac{3}{4}}(\eta\eta'^2)^{-\frac{3}{4}}\exp(\pm\tfrac{2}{3}\varepsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}})H(\eta,\varepsilon) \tag{3.6}$$

where $H(\eta, \varepsilon)$ is a Poincaré series in powers of $\varepsilon^{\frac{3}{2}}$, then we have

$$\bar{\chi}_1 = -\bar{\chi}_-(\eta) \text{ and } \bar{\chi}_2 = i\bar{\chi}_+(\eta). \tag{3.7}$$

In the complete sense of Olver, these expansions are valid away from the turning point in the sectors $|\mathrm{ph}\,\eta| < 2\pi/3$ and $0 > \mathrm{ph}\,\eta > -4\pi/3$, respectively.

The outer expansions $\bar{\phi}_0$ and $\bar{\phi}_1$ have the form

$$\bar{\phi} = \sum_{n=0}^{\infty} \varepsilon^{3n} \bar{\phi}^{(n)}$$

where $\bar{\phi}^{(0)}$ is a solution of a second order equation. In terms of η, power series representations for $\bar{\phi}_0^{(0)}$ and $\bar{\phi}_1^{(0)}$ may be obtained by the method of Frobenius, and, in the complete sense, ϕ_1 is asymptotic to $\bar{\phi}_1^{(0)}$ for $0 > \mathrm{ph}\,\eta > -2\pi/3$. For applications to the boundary value problem, however, it is more convenient to note that, in terms of y, $\bar{\phi}^{(0)}$ satisfies the reduced equation

$$\tfrac{1}{2}(1 - y^2)\bar{\phi}^{(0)''} - y\bar{\phi}^{(0)'} + \lambda\bar{\phi}^{(0)} = 0. \tag{3.8}$$

Solutions of (3.8) in the complex y-plane cut from -1 to $-\infty$ are linear combinations of the Legendre functions $P_\nu(y)$ and $Q_\nu(y)$ with

$$\nu(\nu + 1) = 2\lambda. \tag{3.9}$$

Appropriate linear combinations in the present case are determined by the behavior of ϕ_0 and ϕ_1 as y tends to the turning point $y_1 = -1$. This is somewhat awkward as the Legendre functions themselves are usually normalized relative to $y = +1$. As a result, the linear combinations of $P_\nu(y)$ and $Q_\nu(y)$ obtained in [4] are fairly complicated and cause difficulties when the endpoint y_0 is not at the origin. Considerably more suitable representations may be obtained by exploiting the relations between $P_\nu(y)$, $Q_\nu(y)$, $P_\nu(-y)$, and $Q_\nu(-y)$. In particular,

$$\left.\begin{aligned} &\bar{\phi}_0^{(0)} = P_\nu(-y) \\ &\text{and} \\ &\bar{\phi}_1^{(0)} = -2Q_\nu(-y) + [\log 2 - 2\psi(\nu + 1)]P_\nu(-y) \end{aligned}\right\} \tag{3.10}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

In the boundary-value problem, outer approximations for the four exact solutions may consistently be used at the second boundary point. However, care must be taken as this point lies directly on the Stokes line $\mathrm{ph}\,\eta = 0$ in the complex plane, and, with the exception of ϕ_0, outer expansions of these solutions exhibit the Stokes phenomenon. This particularity effects ϕ_1 and χ_2 as the Stokes line $\mathrm{ph}\,\eta = 0$ bounds the region where these solutions are asymptotic to $\bar{\phi}_1$ and $\bar{\chi}_2$ alone. Lakin and Ng [5] have shown that, in the complete sense, mean expansions should be used on Stokes lines. We thus have

$$\left.\begin{aligned} &\phi_1 \sim \bar{\phi}_1 + \frac{\tau_0}{2}\bar{\phi}_0 + \frac{\tau}{2}\bar{\chi}_1, \\ &\chi_2 \sim \bar{\chi}_2 + \frac{\delta}{2}\bar{\chi}_1 + \frac{\delta_0}{2}\bar{\phi}_0 + \frac{\delta_1}{2}\bar{\phi}_1 \end{aligned}\right\} \tag{3.11}$$

when $\operatorname{ph} \eta = 0$ where $\mathscr{s}$, $\mathscr{s}_0$, $\mathscr{s}_1$, τ_0, and τ are Stokes multipliers and, to order $\varepsilon^3 \log \varepsilon$, $\mathscr{s} = \mathscr{s}_0 = -1$, $\tau_0 = \tau = 2\pi i$, while $\mathscr{s}_1 = O(\varepsilon^3)$. We note that on this Stokes line, the exponentials in $\bar{\chi}_2$ and $\bar{\chi}_1$ are maximally large and small, respectively.

Inner expansions for these four exact solutions involve the stretched variable $\xi = \eta/\varepsilon$ reflecting the fact that the critical layer about the turning point has thickness order ε. The expansion $\tilde{\phi}_0(\xi)$ is simply a power series in ξ, and at the turning point, $\tilde{\phi}_0(0) = 1$. First approximations to $\tilde{\phi}_1(\xi)$, $\tilde{\chi}_1(\xi)$, and $\tilde{\chi}_2(\xi)$ are the generalized Airy functions $B_3(\xi; 1, 1)$, $A_1(\xi, 1)$, and $A_2(\xi, 1)$, respectively. Close to and at the turning point, the four exact solutions have the asymptotic behavior

$$\left.\begin{aligned} &\phi_0 \sim \tilde{\phi}_0(\xi),\ \chi_j \sim D_j(\varepsilon)\tilde{\chi}_j(\xi), \quad j = 1, 2, \\ &\text{and} \\ &\phi_1 \sim C_1(\varepsilon)\tilde{\phi}_1(\xi) + B_1(\varepsilon)\tilde{\phi}_0(\xi) \end{aligned}\right\} \tag{3.12}$$

where $B_1(\varepsilon)$, $C_1(\varepsilon)$, and $D_j(\varepsilon)$ are central matching coefficients. To order ε^3, $B_1 \sim \log \varepsilon$, $C_1 \sim -1$, and $D_j \sim 1$.

4. The characteristic equation

In terms of the Langer variable η, the boundary conditions (3.2) are

$$\phi(\eta_0) = \frac{d\phi(\eta_0)}{d\eta} = \mathscr{B}_2\phi(0) = \mathscr{B}_3\phi(0) = 0 \tag{4.1}$$

where $\eta_0 = \eta(y_0)$, $\mathscr{B}_2$ and $\mathscr{B}_3$ are the differential operators

$$\mathscr{B}_2 = \frac{d^2}{d\eta^2} + \gamma \frac{d}{d\eta},\ \mathscr{B}_3 = \frac{d^3}{d\eta^3} + [\gamma' + \gamma^2] \frac{d}{d\eta}, \tag{4.2}$$

and $\gamma(\eta) = \eta''(y)/(\eta'(y))^2$. As the general solution of (3.2) must be a linear combination of ϕ_0, ϕ_1, χ_1, and χ_2, conditions (4.1) now lead to a characteristic equation involving a four-by-four determinant. Let

$$W(X, Y) = X(\eta_0) \frac{dY}{d\eta}(\eta_0) - \frac{dX}{d\eta}(\eta_0) Y(\eta_0)$$

and $B(X, Y) = \mathscr{B}_3 X(0) \mathscr{B}_2 Y(0) - \mathscr{B}_2 X(0) \mathscr{B}_3 Y(0)$. Then, the exact eigenvalue relation may be written in the form

$$\begin{aligned} &B(\phi_0, \phi_1)W(\chi_1, \chi_2) - B(\phi_0, \chi_1)W(\phi_1, \chi_2) + B(\phi_0, \chi_2)W(\phi_1, \chi_1) \\ &\quad + B(\phi_1, \chi_1)W(\phi_0, \chi_2) - B(\phi_1, \chi_2)W(\phi_0, \chi_1) + B(\chi_1, \chi_2)W(\phi_0, \phi_1) = 0. \end{aligned} \tag{4.3}$$

After using outer approximations to the exact solutions at η_0 and matched inner approximations at the turning point $\xi = 0$, equation (4.3) contains three distinct types of terms and may be written as

$$\mathscr{D} + \mathscr{B} + \mathscr{R} = 0. \tag{4.4}$$

Terms in $\mathscr{D}$ involve a multiple of either $W(\bar{\phi}_0, \bar{\chi}_2)$ or $W(\bar{\phi}_1, \bar{\chi}_2)$ and are dominant as they contain the exponential factor $E(\eta_0) = \exp\{\frac{2}{3}\varepsilon^{-\frac{3}{2}}\eta_0^{\frac{3}{2}}\}$. In particular

$$\mathscr{D} = D_1(\varepsilon)W(\bar{\phi}_0, \bar{\chi}_2)\left[C_1(\varepsilon)B(\tilde{\phi}_1, \tilde{\chi}_1) + \left\{B_1(\varepsilon) - \frac{\tau_0}{2}\right\}B(\tilde{\phi}_0, \tilde{\chi}_1)\right]$$
$$- D_1(\varepsilon)W(\bar{\phi}_1, \bar{\chi}_2)B(\tilde{\phi}_0, \tilde{\chi}_1). \tag{4.5}$$

Terms in $\mathscr{B}$ do not involve exponential factors and are balanced, while terms in $\mathscr{R}$ are recessive as they contain the exponential factor $E^{-1}(\eta_0)$. Because of these exponentials, for small ε we have

$$\mathscr{D} \gg \mathscr{B} \gg \mathscr{R}.$$

It is convenient to explicitly take account of these rapidly varying exponentials and write the characteristic equation as

$$\Delta_1(\alpha, \varepsilon)E(\eta_0) + \Delta_2(\alpha, \varepsilon) + \Delta_3(\alpha, \varepsilon)E^{-1}(\eta_0) = 0 \tag{4.6}$$

The limiting behavior of the eigenvalues $\lambda(\alpha, \varepsilon)$ as ε tends to zero now comes from the first approximation to $\Delta_1(\alpha, \varepsilon)$. Indeed, as η_0 lies directly on the Stokes line $\mathrm{ph}\,\eta = 0$, the exponentials $E(\eta_0)$ and $E^{-1}(\eta_0)$ are maximally large and small, respectively. As a result even for moderate values of ε, both $\Delta_2(\alpha, \varepsilon)$ and $\Delta_3(\alpha, \varepsilon)E^{-1}(\eta_0)$ will be small compared to the error term in $\Delta_1(\alpha, \varepsilon)E(\eta_0)$. The approximate equation

$$\Delta_1(\alpha, \varepsilon) = 0 \tag{4.7}$$

will thus yield good approximations to $\lambda(\alpha, \varepsilon)$ over a wide range of ε.

As $\tilde{\phi}_0$, $\tilde{\phi}_1$, and $\tilde{\chi}_1$ are functions of the stretched variable $\xi = \eta/\varepsilon$ and $d/d\eta = \varepsilon^{-1}\, d/d\xi$, the leading term in the expansions for both $B(\tilde{\phi}_0, \tilde{\chi}_1)$ and $B(\tilde{\phi}_1, \tilde{\chi}_1)$ would seem to be of order ε^{-5}. However, care must be taken as $A_1'''(\xi, 1) = \xi A_1(\xi, 0)$ which vanishes at the turning point, so both $\mathscr{B}_2\tilde{\chi}_1$ and $\mathscr{B}_3\tilde{\chi}_1$ are, in fact, order ε^{-2}. Similarly, both $\mathscr{B}_2\tilde{\phi}_0$ and $\mathscr{B}_3\tilde{\phi}_0$ are order one. Hence, in (4.5), only $B(\tilde{\phi}_1, \tilde{\chi}_1)$ is order ε^{-5} while $B(\tilde{\phi}_0, \tilde{\chi}_1)$ is order ε^{-2}. In addition, $\bar{\chi}_2'$ is larger than $\bar{\chi}_2$ by a factor $\varepsilon^{\frac{3}{2}}$. These facts lead to the expression

$$\Delta_1(\alpha, \varepsilon) = \frac{3^{-\frac{1}{3}}\pi^{-\frac{1}{2}}}{2\Gamma(\frac{1}{3})}\,\varepsilon^{-23/4}\eta^{-\frac{1}{4}}\eta'^{-\frac{3}{2}}\bar{\phi}_0^{(0)}(\eta_0)\mathrm{e}^{-\pi i/6}\{1 + O(\varepsilon^{\frac{3}{2}})\}. \tag{4.8}$$

By (4.7) and (3.10), first approximations to the eigenvalues are thus determined by values of ν for which the Legendre function $P_\nu\{\alpha/(\alpha - 1)\}$ vanishes. In particular, if $\nu_n(\alpha)$ denotes the n-th positive zero of this function for fixed α, we have

$$\lambda_n(\alpha, \varepsilon) = \tfrac{1}{2}\nu_n(\alpha)[\nu_n(\alpha) + 1] + O(\varepsilon^{\frac{3}{2}}). \tag{4.9}$$

In the hub-clamped case when $\alpha = 0$, $P_\nu(0)$ is just a multiple of $\cos(\nu\pi/2)$. Hence, in agreement with [4], $\nu_n(0)$ is simply an odd positive integer.

To examine the present situation when α is non-zero, let the angle θ_0 in radians be defined by

$$\cos\theta_0 = \frac{\alpha}{\alpha - 1}. \tag{4.10}$$

Hence, $0 < \alpha < \pi/2$ corresponds to $\pi/2 < \theta_0 < \pi$. For fixed values of θ_0 in this range, the asymptotic behavior of $P_\nu(\cos\theta_0)$ may now be used to obtain explicit analytic formulas for the large zeros ν_n. In particular,

$$P_\nu(\cos\theta_0) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\left(\frac{\pi}{2}\sin\theta_0\right)^{-\frac{1}{2}}\cos\left[(\nu+\tfrac{1}{2})\theta_0 - \frac{\pi}{4}\right] + O(\nu^{-1}). \tag{4.11}$$

Thus, if $\beta = \theta_0/\pi$, $P_\nu(\cos\theta_0) = 0$ implies that for large n

$$\nu_n(\alpha) \sim \frac{1}{\beta}\left\{\frac{4n+3}{4}\right\} - \tfrac{1}{2}. \tag{4.12}$$

The higher eigenvalues now have the asymptotic forms

$$\lambda_n(\alpha) = \frac{1}{8}\left\{\left(\frac{4n+3}{2\beta}\right)^2 - 1\right\} + O\left(\varepsilon^{\frac{3}{2}}, \frac{1}{n}\right). \tag{4.13}$$

Unfortunately, when θ_0 is simply between $\pi/2$ and π but n is not large, the trigonometric expansion for $P_\nu(\cos\theta_0)$ does not admit an explicit formula for ν_n and, hence, the lower eigenvalues. Unless θ_0 is close to $\pi/2$, zeros of the Legendre function must now be obtained numerically. This type of problem arises in the study of diffraction of waves by a cone where the desired information is the zeros of the associated Legendre function $P_\nu^1(\cos\theta_0)$

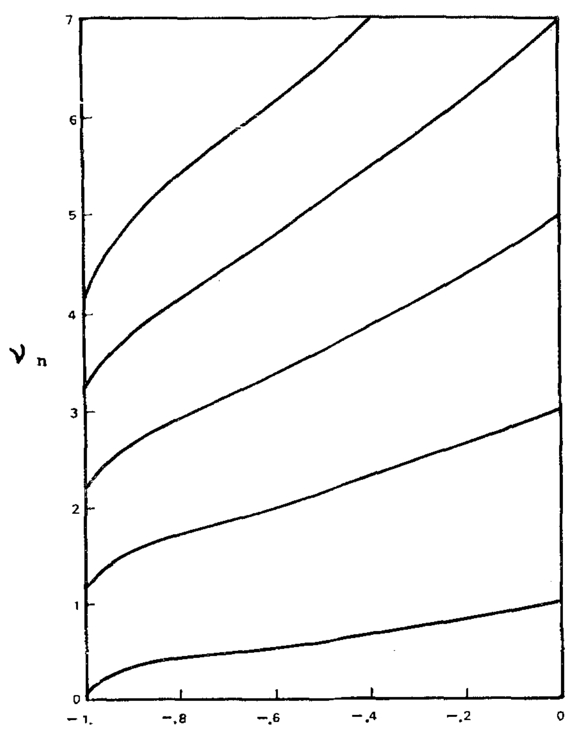

Figure 2. Zeroes of the Legendre function $P_\nu(\cos\theta_0)$.

for fixed θ_0. Surprisingly little work has been done for the Legendre functions themselves, however. Whereas the first fifty zeros of $P_\nu^1(\cos\theta_0)$ have been computed to high accuracy by several authors, only the first five zeros of $P_\nu(\cos\theta_0)$ have been computed by Hall [3]. Results for ν_n as a function of $\cos\theta_0$ with $n = 1$ to 5 are shown in Figure 2.

Finally, we wish to examine the lower eigenvalues of the wobbling hub-clamped rod for which θ_0 is close to $\pi/2$, i.e. α is close to zero. In this case, expanding the Legendre function about $\alpha = 0$ gives the relation

$$\cot\left[\frac{\nu(\alpha)\pi}{2}\right] = \frac{2\alpha}{1-\alpha} + O(\alpha^2)$$

and hence, for $n = 1, 2, 3, \ldots$

$$\nu_n(\alpha) = 2n - 1 - \frac{4\alpha}{\pi(1-\alpha)} + O(\alpha^2). \tag{4.14}$$

For small α, we therefore have

$$\lambda_n(\alpha) = n(2n-1) - \frac{2\alpha(4n-1)}{\pi(1-\alpha)} + O(\varepsilon^{\frac{3}{2}}, \alpha^2), \qquad n = 1, 2, 3, \ldots. \tag{4.15}$$

A wobble will thus lower the natural frequencies from the values of the corresponding hub-clamped rod, with the effect being more pronounced as n increases.

Acknowledgment

This research has been supported in part by the National Research Council of Canada under grant number A7850.

REFERENCES

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C. (1964).
[2] W. E. Boyce, R. C. DiPrima and G. H. Handelman, Vibrations of rotating beams of constant section, *Proc. Second U.S. Nat. Cong. Appl. Mech.*, (1954) 165–173.
[3] R. N. Hall, The application of non-integral Legendre functions to potential problems, *J. Appl. Phys.*, 20 (1949) 925–931.
[4] W. D. Lakin, On the differential equation of a rapidly rotating slender rod, *Quart. Appl. Math.*, 32 (1974) 11–27.
[5] W. D. Lakin and B. S. Ng, A fourth order eigenvalue problem with a turning point at the boundary, *Quart. J. Mech. Appl. Math.*, 28 (1975) 107–121.
[6] J. C. P. Miller, On the choice of standard solutions for a homogeneous linear differential equation of the second order, *Quart. J. Mech. Appl. Math.* 3 (1950) 225–235.
[7] N. Mostaghel and I. Tadjbakhsh, Buckling of rotating rods and plates, *Int. J. Mech. Sci.*, 15 (1973) 429–434.
[8] A. Nachman, The buckling of rotating rods, *J. Appl. Mech.* 42 (1975) 222–224.